CONFORMALLY INVARIANT QUANTIZATION – TOWARDS COMPLETE CLASSIFICATION

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Abstract. Let M be a smooth manifold equipped with a conformal structure, $\mathcal{E}[w]$ the space of densities with the the conformal weight w and $\mathcal{D}_{w,w+\delta}$ the space of differential operators from $\mathcal{E}[w]$ to $\mathcal{E}[w+\delta]$. Conformal quantization Q is a right inverse of the principle symbol map on $\mathcal{D}_{w,w+\delta}$ such that Q is conformally invariant and exists for all w. This is known to exists for generic values of δ. We give explicit formulae for Q for all δ out of the set of critical weights Σ. We provide a simple description of this set and conjecture its minimality.

1. Introduction

The notion of quantization originates in physics. Here we view it as quest for a correspondence between a space of differential operators and the corresponding space of symbols. More specifically, consider the space $\mathcal{D}_0$ of differential operators acting on smooth functions on a smooth manifold M and the space of symbols $\mathcal{S}_0$. Quantization is a map $Q_0 : \mathcal{S}_0 \to \mathcal{D}_0$ such that $\operatorname{Symb} \circ Q_0 = \operatorname{id}|_{\mathcal{S}_0}$ where $\operatorname{Symb} : \mathcal{D}_0 \to \mathcal{S}_0$ is the principal symbol map. If $\Phi \in \mathcal{D}_0$ of the order k has the principal symbol σ then $\Phi - Q_0(\sigma) \in \mathcal{D}_0$ has the order $k-1$. Iterating this we obtain the isomorphism of vector spaces $\bigoplus_{i=0}^{k} \mathcal{S}_0^i \cong \mathcal{D}_0^k$ where $\mathcal{S}_0^i = \Gamma(\bigodot^i TM) \subseteq \mathcal{S}_0$ and $\mathcal{D}_0^k \subseteq \mathcal{D}_0$ is the space of operators of order at most k. Here $\bigodot^k$ is the kth symmetric tensor product. We shall use the notation $Q_0^\sigma := Q_0(\sigma)$.

There is no natural quantization on a M. On the other hand, e.g. a choice of a linear connection ∇ on M yields a prefered quantization in an obvious way: if $\sigma \in \mathcal{S}_0^k$ and $f \in C^\infty(M)$, we put $Q_0^\sigma(f) = \sigma(\nabla^{(k)} f)$ where $\nabla^{(k)} f$ is the symmetrized k–fold covariant derivative. Therefore there is a canonical quantization on every pseudo-Riemannian manifold M. Motivated by this observation one can ask whether there is a natural quantization for less rigid geometrical structures on M.

In this article we study the case when the manifold M is equipped with a conformal structure. This was iniciated by Duval, Lecomte and Ovsienko [12], see also [24] for the projective case. The study of quantization for these (and related) structures has been very active in recent years, we refer to the survey [25] and references therin for the state of art.

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The conformal structure on manifold M is a class of pseudo–Riemannian metrics $[g] = \{fg \mid f \in \mathbb{C}^\infty(M), f > 0\}$ on a manifold M. The homegeneous model is the pseudosphere $M = S^{p,q} := S^p \times S^q$, where (p,q) is the signature of g, the product of the standard metrics on S^p and S^q. This is homogeneous space for $G = SO_0(p+1, q+1)$ acting on $S^{p,q}$ by conformal motions of $[g]$ and we have the isomorphism $S^{p,q} \cong G/P$ where $P \subseteq G$ is the Poincare conformal group of motions fixing a point, see [8] for details. Then both $\mathcal{S}_0$ and $\mathcal{D}_0$ are G–modules and the question of conformally invariant quantization means to construct $Q_0 : \mathcal{S}_0 \to \mathcal{D}_0$ which intertwines these G–actions. If we pass from $S^{p,q}$ to $\mathbb{R}^{p,q}$ via the stereographic projection, we replace the G–action (which is not defined on $\mathbb{R}^{p,q}$) by the infinitesimal $\mathfrak{g}$–action. The Lie algebra $\mathfrak{g}$ of G can be realized as a Lie algebra of (polynomial) vector fields on $\mathbb{R}^{p,q}$ and they act by the Lie derivative as infinitesimal conformal symmetries. The same can be done for every locally conformally flat manifold and the invariance of Q_0 is given by this $\mathfrak{g}$–action. This setting is often taken as the starting point in the study of invariant (or equivariant) quantization [12]. It is natural to consider more generally bundles of conformal densities $E[w]$, $w \in \mathbb{R}$ (instead just functions) and the space of differential operators $\Gamma(E[w_1]) \to \Gamma(E[w_2])$ denoted by $\mathcal{D}_{w_1,w_2}$. Denoting by $\mathcal{D}^k_{w_1,w_2}$ the space of operators of degree $\leq k$, the corresponding bundle of kth degree symbols is then $S^k_\delta = (\bigodot^k TM) \otimes E[\delta] \cong \mathcal{D}^k_{w_1,w_2}/\mathcal{D}^{k-1}_{w_1,w_2}$ where $\delta = w_2 - w_1$. Note this is the notation used in the conformally invariant calculus; the space of densities can be also defined as $\mathcal{F}_\lambda = \Gamma(\otimes^\lambda(\bigwedge^n T^*M))$ where $\bigwedge^n T^*M \to M$ is the determinant bundle, $n = \dim(M)$. Then one has the relation $\Gamma(E[-nw]) = \mathcal{F}_w$.

Summarizing, the question in the conformally flat case is whether for a given $\delta \in \mathbb{R}$ there is an isomorphism of $\mathfrak{so}_{p+1,q+1}$-modules

$$Q_\delta : \mathcal{S}_\delta \longrightarrow \mathcal{D}_{w,w+\delta} \tag{1}$$

for all $w \in \mathbb{R}$ where $S_\delta = (\bigodot TM) \otimes E[\delta]$. That is, the corresponding bilinear differential operator $Q_\delta : \mathcal{S}_\delta \times \mathcal{E}[w] \to \mathcal{E}[w+\delta]$ is conformally invariant. It turns out the answer is positive for a generic weight δ. More precisely, it is shown in [12] that if $\delta \notin \widetilde{\Sigma}$ where $\widetilde{\Sigma}$ is the set of *critical weights* from [12] then the conformal quantization Q_δ exists. Note to get a complete answer one needs to study critical weights for particular irreducible components of $\mathcal{S}_\delta$.

Now we turn to the curved case where M is a manifold with the given conformal class $[g]$. Then there are generically no infinitesimal symmetries on $(M,[g])$ and by invariance of the quantization $Q_\delta : \mathcal{S}_\delta \to \mathcal{D}_{w,w+\delta}$ we mean the corresponding bilinear operator $Q_\delta : \mathcal{S}_\delta \times \mathcal{E}[w] \to \mathcal{E}[w+\delta]$ is given in terms of a Levi–Civita connection ∇ from the conformal class, its curvature R and algebraic operations in such a way that Q_δ does not depend on the choice of ∇. (This is equivalent to the $\mathfrak{so}_{p+1,q+1}$-invariance on conformally flat manifolds [8].) Using the terminology of conformal geometry, Q_δ has a *curved analogue*. Note there is generally no hope for uniqueness of Q_δ as the curvature can modify conformal operators in various ways.

Let us briefly summarize the development iniciated by [12] where the conformally flat case is considered. On one hand, there are several results for lower order

cases [13, 11, 26]. On the other hand, in the recent Kroeske's thesis [23], a general problem of construction of conformal bilinear operators $V_1 \times V_2 \to W$ for given irreducible conformal bundles V_1, V_2 and W is solved provided conformal weights of the bundles concerned are not critical. In fact, the result in [23] is much stronger as it provides such construction for the wide class of *parabolic geometries*. Conformal geometry is the most studied parabolic structure, other parabolic geometries are e.g. projective, contact projective or CR. In particular, parabolic geometries cover all "IFFT–cases" [3]. For conformal structures, the case Q_0 is related to construction of symmetries of differential operators, see e.g. [14, 16] for the Laplace operator. The construction in [23] is very general however it is clear how to obtain quantization from the machinery developed there. (The question of symbols and possible dependence on w is not explicitly addressed there).

To classify the conformal quantization, the two basic quastions are the minimality of the critical set Σ in the flat case and existence (and explicit construction) of Q_δ for $\delta \notin \Sigma$ in the curved case. There are (up to our knowledge) no nonexistence results for critical conformal cases on $S^{p,q}$ hence the minimality is an issue. ($\widetilde{\Sigma}$ from [12] is not minimal as observed in [11] for the third order quantization.) An explicit construction for curved conformal manifolds is known only trace–free symbols in $\mathcal{S}_\delta$ [29].

Here we focus on the construction but also obtain a partial step towards minimality of the critical set. The main result is Theorem 3.3 which provides an explicit (and inductive) formula for Q_δ on all curved conformal manifolds. We obtain the critical set Σ which is smaller than corresponding sets in [12] or [23] and agrees with [11] for the order three. Moreover, we indicate some reasons why our set of critical weights Σ should be minimal in Proposition 4.2. We shall discuss minimality of this set in the follow up work [30] in detail.

Let us comment upon what we mean by explicit construction. There is obviously no reason to ask for a formula in terms of a Levi–Civita connection ∇ (and its curvature) from the conformal class. These are getting extremely complicated already for higher order *linear* conformal operators [21]. The conformal analogue of Riemannian ∇–calculus is the *tractor calculus*, see [1] for a discussion on its origin. It is closely related to the Cartan connection [7, 6] and can be viewed as a linear or "explicit" version of the Cartan connection. The transformation from tractors to formulae in terms of Levi–Civita connection is given by simple rules, see [21] for a computer implementation. In Theorem 3.3 we obtain simple tractor formulae for the conformal quantization Q_δ. Then we discuss the critical set Σ in details and conjecture its minimality, see Section 4.

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2. Conformal geometry and tractor calculus

2.1. **Notation and background.** We present here a brief summary, further details may be found in [5, 21]. Let M be a smooth manifold of dimension $n \geq 3$. Recall that a *conformal structure* of signature (p, q) on M is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$ whose fibre over x consists of conformally related signature-(p, q)

metrics at the point x. Sections of $\mathcal{Q}$ are metrics g on M. So we may equivalently view the conformal structure as the equivalence class $[g]$ of these conformally related metrics. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$ induces a natural line bundle on $(M, [g])$ that we term the conformal density bundle $E[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. We write $\mathcal{E}^a$ for the space of sections of the tangent bundle TM and $\mathcal{E}_a$ for the space of sections of T^*M. The indices here are abstract in the sense of [28] and we follow the usual conventions from that source. So for example $\mathcal{E}_{ab}$ is the space of sections of $\otimes^2 T^*M$. Here and throughout, sections, tensors, and functions are always smooth. When no confusion is likely to arise, we will use the same notation for a bundle and its section space.

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $S^2T^*M \otimes E[2]$ determined by the conformal structure. This is used to identify TM with $T^*M[2]$. For many calculations we employ abstract indices in an obvious way. Given a choice of metric g from $[g]$, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Here we are raising indices and contracting using the (inverse) conformal metric. Indices will be raised and lowered in this way without further comment. Note $E[w]$ is trivialized by a choice of metric g from the conformal class, and we also write ∇ for the connection corresponding to this trivialization. The coupled ∇_a preserves the conformal metric.

The curvature $R_{ab}{}^c{}_d$ of the Levi-Civita connection (the Riemannian curvature) is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d$ ($[\cdot,\cdot]$ indicates the commutator bracket). This can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to

$$R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}P_{b]d} + 2\boldsymbol{g}_{d[b}P_{a]c}, \tag{2}$$

where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. The Schouten tensor is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^c{}_b$ and vice versa: $\mathrm{Ric}_{ab} = (n-2)P_{ab} + J\boldsymbol{g}_{ab}$, where we write J for the trace $P_a{}^a$ of P. The *Cotton tensor* is defined by $A_{abc} := 2\nabla_{[b}P_{c]a}$. Via the Bianchi identity this is related to the divergence of the Weyl tensor as follows:

$$(n-3)A_{abc} = \nabla^d C_{dabc}. \tag{3}$$

Under a *conformal transformation* we replace a choice of metric g by the metric $\hat{g} = e^{2\Upsilon}g$, where Υ is a smooth function. We recall that, in particular, the Weyl curvature is conformally invariant $\widehat{C}_{abcd} = C_{abcd}$. With $\Upsilon_a := \nabla_a\Upsilon$, the Schouten tensor transforms according to

$$\widehat{P}_{ab} = P_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon^c\Upsilon_c\boldsymbol{g}_{ab}. \tag{4}$$

Explicit formulae for the corresponding transformation of the Levi-Civita connection and its curvatures are given in e.g. [1, 21]. From these, one can easily compute the transformation for a general valence (i.e. rank) s section $f_{bc\cdots d} \in \mathcal{E}_{bc\cdots d}[w]$

using the Leibniz rule:

$$\begin{aligned}\hat{\nabla}_{\bar{a}} f_{bc\cdots d} =& \nabla_{\bar{a}} f_{bc\cdots d} + (w-s)\Upsilon_{\bar{a}} f_{bc\cdots d} - \Upsilon_b f_{\bar{a}c\cdots d} \cdots - \Upsilon_d f_{bc\cdots \bar{a}} \\ &+ \Upsilon^p f_{pc\cdots d}\boldsymbol{g}_{b\bar{a}} \cdots + \Upsilon^p f_{bc\cdots p}\boldsymbol{g}_{d\bar{a}}.\end{aligned} \tag{5}$$

We next define the standard tractor bundle over $(M,[g])$. It is a vector bundle of rank $n+2$ defined, for each $g \in [g]$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon} g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$ by the transformation

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\mu}_a \\ \widehat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix}. \tag{6}$$

It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{E}^A$ over the conformal manifold. (Alternatively the standard tractor bundle may be constructed as a canonical quotient of a certain 2-jet bundle or as an associated bundle to the normal conformal Cartan bundle [6].) On a conformal structure of signature (p,q), the bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(p+1,q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. Up up to isomorphism this the unique *normal conformal tractor connection* [7] and it induces a normal connection on $\bigotimes \mathcal{E}^A$ that we will also denoted by ∇_a and term the (normal) tractor connection. In a conformal scale g, the metric h_{AB} and ∇_a on $\mathcal{E}^A$ are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + P_{ab}\alpha \\ \nabla_a \tau - P_{ab}\mu^b \end{pmatrix}. \tag{7}$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we see that

$$Y_A X^A = 1, \quad Z_{Ab} Z^A{}_c = \boldsymbol{g}_{bc}, \tag{8}$$

and all other quadratic combinations that contract the tractor index vanish. In (6) note that $\widehat{\alpha} = \alpha$ and hence X^A is conformally invariant.

The curvature Ω of the tractor connection is defined on $\mathcal{E}^C$ by $[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E$. Using (7) and the formulae for the Riemannian curvature yields

$$\Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - 2X_{[C} Z_{E]}{}^e A_{eab} \tag{9}$$

Given a choice of $g \in [g]$, the *tractor-D operator* $D_A\colon \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$ is defined by

$$D_A V := (n+2w-2)wY_A V + (n+2w-2)Z_{Aa}\nabla^a V - X_A \Box V, \tag{10}$$

where $\Box V := \Delta V + wJV$. This is conformally invariant, as can be checked directly using the formulae above (or alternatively there are conformally invariant constructions of D, see e.g. [18]).

The operator D_A is strongly invariant. That is, it is invariant as an operator

$$D_A : \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$$

where now we interpret ∇ in (10) as the couple Levi–Civita–tractor connection. Note the strong invariance is a property of a *formulae*, see [19, p.21] for a more detailed discussion and [15, (2)] for an illustrative example. We shall say an operator is strongly invariant if it is clear which formula we mean. Note composition of two strongly invariant operators is strongly invariant.

2.2. **Tractor connection and standard tractors.** Using the standard tractors X_B, Z^b_B and Y_B, the tractor connections takes the form

$$\begin{aligned}
\nabla_a Y_B\sigma &= Y_B\nabla_a\sigma + Z^b_B P_{ab}\sigma, && \sigma \in \mathcal{E}[w]\\
\nabla_a Z^b_B\mu_b &= -Y_B\mu_a + Z^b_B\nabla_a\mu_b - X_B P_a{}^b\mu_b, && \mu_b \in \mathcal{E}_b[w]\\
\nabla_a X_B\rho &= Z^b_B \boldsymbol{g}_{ab}\rho + X_B\nabla_a\rho, && \rho \in \mathcal{E}[w]
\end{aligned} \tag{11}$$

which follows from (7) (or see e.g. [21]). More accurately, ∇ denotes the coupled tractor–Levi-Civita connection in expressions like in the previous display.

We shall need, more generally, to know how the composition of several applications of the tractor connection acts on standard tractors. In fact, we shall need this only on $\mathbb{R}^{p,q}$. It follows from (11) (and can be verified easily by induction wrt. $k \geq 1$) that

$$\begin{aligned}
\nabla_{(a_1}\ldots\nabla_{a_k)}Y_B\sigma =& Y_B\nabla_{(a_1}\ldots\nabla_{a_k)}\sigma + ct,\\
\nabla_{(a_1}\ldots\nabla_{a_k)}Z^b_B\mu_b =& -kY_B\delta^b_{(a_1}\nabla_{a_2}\ldots\nabla_{a_k)}\mu_b + Z^b_B\nabla_{(a_1}\ldots\nabla_{a_k)}\mu_b + ct,\\
\nabla_{(a_1}\ldots\nabla_{a_k)}X_B\rho =& -\frac{1}{2}k(k-1)Y_B\boldsymbol{g}_{(a_1a_2}\nabla_{a_3}\ldots\nabla_{a_k)}\rho + kZ^b_B\boldsymbol{g}_{b(a_1}\nabla_{a_2}\ldots\nabla_{a_k)}\rho\\
&+ X_B\nabla_{(a_1}\ldots\nabla_{a_k)}\rho + ct
\end{aligned}$$

where $\sigma \in \mathcal{E}[w]$, $\mu_b \in \mathcal{E}_b[w]$, $\rho \in \mathcal{E}[w]$ and "ct" denotes terms which involve curvature and at most $k-2$ derivatives. (That is, "ct" vanishes on $\mathbb{R}^{p,q}$.) Here and below, $(\ldots)$ denotes symmetrization of the enclosed indices and the notation $(\ldots)_0$ will denote the projection to the symmetric trace-free part. In fact, the previous display holds also for $k=0$ if we consider expressions with k free indices $a_1\cdots a_k$ simply being absent for $k=0$. Henceforth we shall use this convention. It follows from the previous display (or can be verified by induction directly) that for $k \geq 0$

we obtain

$$
\begin{aligned}
&\nabla_{(a_1}\dots\nabla_{a_k)_0} Y_B\sigma = Y_B\nabla_{(a_1}\dots\nabla_{a_k)_0}\sigma + ct,\\
(12)\quad &\nabla_{(a_1}\dots\nabla_{a_k)_0} Z_B^b\mu_b = -\,kY_B\delta^b_{(a_1}\nabla_{a_2}\dots\nabla_{a_k)_0}\mu_b + Z_B^b\nabla_{(a_1}\dots\nabla_{a_k)_0}\mu_b + ct,\\
&\nabla_{(a_1}\dots\nabla_{a_k)_0} X_B\rho = kZ_B^b\boldsymbol{g}_{b(a_1}\nabla_{a_2}\dots\nabla_{a_k)_0}\rho + X_B\nabla_{(a_1}\dots\nabla_{a_k)_0}\rho + ct
\end{aligned}
$$

and for $\ell \geq 0$ we have

$$
\begin{aligned}
&\Delta^\ell Y_B\sigma = Y_B\Delta^\ell\sigma + ct,\\
(13)\quad &\Delta^\ell Z_B^b\mu_b = -\,2\ell Y_B\nabla^b\Delta^{\ell-1}\mu_b + Z_B^b\Delta^\ell\mu_b + ct,\\
&\Delta^\ell X_B\rho = -\,\ell(n+2\ell-2)Y_B\Delta^{\ell-1}\rho + 2\ell Z_B^b\nabla_b\Delta^{\ell-1}\rho + X_B\Delta^\ell\rho + ct
\end{aligned}
$$

where $\sigma \in \mathcal{E}[w]$, $\mu_b \in \mathcal{E}_b[w]$, $\rho \in \mathcal{E}[w]$.

3. Tractor construction of conformal quantization and critical weights

We assume $\sigma^{a_1\dots a_k} \in \mathcal{E}^{(a_1\dots a_k)}[\delta] =: \mathcal{S}_{\delta,k}$ and $f \in \mathcal{E}[w]$. Our aim is to construct a quantization i.e. a differential operator $Q^\sigma_\delta : \mathcal{E}[w] \to \mathcal{E}[w+\delta]$ with the leading term $\sigma^{a_1\dots a_k}\nabla_{a_1}\cdots\nabla_{a_k}$. The bundle of symbols $\mathcal{E}^{(a_1\dots a_k)}[\delta]$ decomposes into irreducibles as

$$\mathcal{E}^{(a_1\dots a_k)}[\delta] = \bigoplus_{i=0}^{\lfloor \frac{k}{2}\rfloor} \mathcal{E}^{(a_1\dots a_{k-2i})_0}[\delta+2i]$$

where $\lfloor a \rfloor$ denotes the lower integer part of $a \in \mathbb{R}$. We can assume σ is irreducible (as Q^σ_δ is linear in σ) so

$$
\begin{gathered}
\sigma^{a_1\dots a_k} = \sigma'^{(a_1\dots a_{k'}}\boldsymbol{g}^{a_{k'+1}a_{k'+2}}\dots\boldsymbol{g}^{a_{k'+2\ell-1}a_{k'+2\ell})}, \quad k'+2\ell = k \quad \text{where}\\
(\sigma')^{a_1\dots a_{k'}} \in \mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta'], \quad \delta' = \delta + 2\ell
\end{gathered}
$$

since $\boldsymbol{g}^{ab} \in \mathcal{E}^{ab}[-2]$.

Henceforth we consider the irreducible symbol σ' as in the previous display. Our aim is to construct a differential operator

$$
(14)\qquad
\begin{aligned}
&Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w+\delta'-2l]\\
&Q^{\sigma'}_{k',\ell}(f) = (\sigma')^{a_1\dots a_{k'}}\nabla_{(a_1}\dots\nabla_{a_{k'})_0}\Delta^\ell f + lot
\end{aligned}
$$

which is conformally invariant as the bilinear operator $Q_{k',\ell} : \mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta'] \times \mathcal{E}[w] \to \mathcal{E}[w+\delta'-2l]$. Here "lot" denotes lower order terms and we have suppressed the parameter δ' in the notation for Q. The reason is that we will define the operator $Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w+\delta'-2l]$ by a universal tractor formula for all $\delta' \in \mathbb{R}$. Then we shall discuss when (i.e. for which δ') $Q^{\sigma'}_{k',\ell}$ fails to have the required leading term.

The construction of $Q_{k',\ell}$ is divided into two steps – the cases $\ell = 0$ and $\ell > 0$.

3.1. **The quantization** $Q_{k',0}$**.** This case is more or less known. Here we shall formulate it as follows.

Theorem 3.1. *Let* $(\sigma')^{a_1\dots a_{k'}} \in \mathcal{E}^{(a_1\dots a_{k'})_0}[\delta']$*. There is an explicit formula for the quantization* $Q^{\sigma'}_{k',0} : \mathcal{E}[w] \to \mathcal{E}[w+\delta']$ *with the leading term* $(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}$ *for every weight* $\delta' \in \mathbb{R}$ *satisfying*

$$\delta' \notin \Sigma_{k',0} \quad \textit{where} \quad \Sigma_{k',0} = \begin{cases} \{-(n+k'+i-2) \mid i = 1,\dots,k'\} & k' \geq 1 \\ \emptyset & k' = 0. \end{cases} \tag{15}$$

Moreover, $Q_{k',0}$ *is strongly conformally invariant in the following sense: if we replace* $f \in \mathcal{E}[w]$ *by* $f \in \mathcal{T} \otimes \mathcal{E}[w]$ *for any tractor bundle* $\mathcal{T}$ *and, in the formula for* $Q_{k',0}$*, we replace the Levi–Civita connection acting on* f *by the coupled Levi–Civita–tractor connection then* $Q_{k',0}$ *is a conformally invariant bilinear operator* $\mathcal{E}^{(a_1\dots a_{k'})_0}[\delta'] \times \mathcal{T} \otimes \mathcal{E}[w] \to \mathcal{T} \otimes \mathcal{E}[w+\delta']$*.*

Remark. The conformal quantization for the case $Q_{k',0}$ was constructed recently in [29] but the strong invariance of this result is unclear. To clarify this point and to keep our presentation self–content, we present the complete proof here.

Proof. We shall use certain splitting operators from $\mathcal{E}[w]$ and $\mathcal{E}^{(a_1\dots a_{k'})_0}[\delta']$ into symmetric tensor products of the adjoint tractor bundle $\mathcal{E}_{[A^1A^2]}$ and their subquotients. To simplify the notation, we shall introduce adjoint tractor indices $\mathbf{A} := [A^1A^2]$. These are just abstract indices of the adjoint tractor bundle. We shall use the notation $f_{(\mathbf{AB})} = \frac{1}{2}(f_{\mathbf{AB}} + f_{\mathbf{BA}})$, $f_{\mathbf{AB}} \in \mathcal{E}_{\mathbf{AB}}$ for the symmetrization, the symmetric tensor products of the adjoint tractor bundle will be denoted by $\mathcal{E}_{(\mathbf{A_1}\dots\mathbf{A_k})}$. Let us note this notation means symmetrization over adjoint indices (not *not* over standard tractor indices), i.e. $f_{(\mathbf{AB})_\mathbf{0}} \neq 0$. The completely trace free component with respect to h^{AB} will be denoted by $\mathcal{E}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}}$. Note the latter bundles are generally not irreducible tractor bundles.

The skew symmetrization with the tractor $X_{A^0_i}$ defines bundle maps $\mathcal{E}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} \to \mathcal{E}_{\mathbf{A_1}\dots[\mathbf{A^0_i}\mathbf{A_i}]\dots\mathbf{A_k}}$. The joint kernel of all these maps for $i = 1,\dots,k$ will be denoted by $\bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}}$. Using the complement $\bar{\mathcal{E}}^{\perp}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} \subseteq \mathcal{E}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}}$ (via the tractor metric h), we obtain the quotient bundle $\tilde{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} := \mathcal{E}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} / \bar{\mathcal{E}}^{\perp}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}}$. One can easily see that choosing a metric from the conformal class, sections of the these have the form

$$\bar{F}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} = \sum_{i=0}^{k} \mathbb{X}^{a_1}_{\mathbf{A_1}} \dots \mathbb{X}^{a_i}_{\mathbf{A_i}} \mathbb{W}_{\mathbf{A_{i+1}}} \dots \mathbb{W}_{\mathbf{A_k}} \bar{f}^i_{(a_1\dots a_i)_0} \quad \text{for} \quad \bar{F}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} \in \bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}},$$

$$\tilde{F}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} = \sum_{i=0}^{k} \mathbb{Y}^{a_1}_{\mathbf{A_1}} \dots \mathbb{Y}^{a_i}_{\mathbf{A_i}} \mathbb{W}_{\mathbf{A_{i+1}}} \dots \mathbb{W}_{\mathbf{A_k}} \tilde{f}^i_{(a_1\dots a_i)_0} \quad \text{for} \quad \tilde{F}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} \in \tilde{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}}$$

for some sections $\bar{f}^i_{(a_1\dots a_i)_0}$ and $\tilde{f}^i_{(a_1\dots a_i)_0}$. Note i is *not* an abstract index here. This describes the composition series for $\bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}}$ and $\tilde{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}}$. (In particular, choosing a metric in the conformal class, both these bundles decompose to exactly $k+1$ irreducible components, e.g. $\bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_\mathbf{0}} = \mathcal{E} \oplus \mathcal{E}_{a_1} \oplus \dots \oplus \mathcal{E}_{(a_1\dots a_k)_0}$.) Also note

the latter bundle is the dual of the former one. Finally, taking the tensor product with density bundles, we obtain $\bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_0}[w]$ and $\tilde{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_0}[w]$ for any $w \in \mathbb{R}$.

Assume $k' \geq 1$. We shall start with $f \in \mathcal{E}[w]$. For an arbitrary chosen metric from the conformal class, we consider the inclusion

$$\bar{\iota} : \mathcal{E}[w] \hookrightarrow \bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_{k'}})_0}[w], \quad f \overset{\bar{\iota}}{\mapsto} \mathbb{W}_{\mathbf{A_1}} \cdots \mathbb{W}_{\mathbf{A_{k'}}} f.$$

Now $\bar{\iota}(f)$ can be extended to a conformally invariant section $\bar{F}_{\mathbf{A_1}\dots\mathbf{A_{k'}}} \in \bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_{k'}})_0}[w]$ as follows: we put $\bar{F}_{\mathbf{A_1}\dots\mathbf{A_{k'}}} := \bar{P}(\mathcal{C})(\bar{\iota}(f)_{A_1\dots A_{k'}})$ where the operator $\bar{P}(\mathcal{C})$ is a suitable polynomial in the *curved Casimir* $\mathcal{C} : \bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_{k'}})_0}[w] \to \bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_{k'}})_0}[w]$ [10]. It follows from the composition series for $\bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_{k'}})_0}[w]$ that the degree of the polynomial $\bar{P}$ is k'. Let us compute the highest order term of $\bar{P}(\mathcal{C})\big(\mathbb{W}_{\mathbf{A_1}} \cdots \mathbb{W}_{\mathbf{A_{k'}}} f\big)$. For this it is sufficient to work on $\mathbb{R}^n$ with the standard metric. Then if P is a polynomial of degree r, $0 \leq r \leq k'$ then there is a (degree r) polynomial p such that $P(\mathcal{C})\big(\mathbb{W}_{\mathbf{A_1}} \cdots \mathbb{W}_{\mathbf{A_{k'}}} f\big) = \mathbb{W}_{\mathbf{A_1}} \cdots \mathbb{W}_{\mathbf{A_{k'}}} p(w) f + \ldots + \mathbb{X}^{\ a_1}_{(\mathbf{A_1}} \ldots \mathbb{X}^{\ a_r}_{\mathbf{A_r}} \mathbb{W}_{\mathbf{A_{r+1}}} \cdots \mathbb{W}_{\mathbf{A_{k'}})} \nabla_{(a_1} \cdots \nabla_{a_r)_0} f$ up to a (nonzero) scalar multiple. This can be easily verified by the induction. Putting $r := k'$, there is a k'–order polynomial $\bar{p}(w)$ such that

$$\bar{F}_{\mathbf{A_1}\dots\mathbf{A_{k'}}} = \mathbb{W}_{\mathbf{A_1}} \cdots \mathbb{W}_{\mathbf{A_{k'}}} \bar{p}(w) f + \ldots + \mathbb{X}^{\ a_1}_{(\mathbf{A_1}} \ldots \mathbb{X}^{\ a_{k'}}_{\mathbf{A_{k'}})} \nabla_{(a_1} \cdots \nabla_{a_{k'})_0} f \tag{16}$$

up to a nonzero scalar multiple.

The splitting for $(\sigma')^{a_1\dots a_{k'}} \in \mathcal{E}^{(a_1\dots a_{k'})_0}[\delta']$ is analogous. We shall start with the inclusion

$$\tilde{\iota} : \mathcal{E}^{(a_1\dots a_{k'})_0}[\delta'] \hookrightarrow \bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_{k'}})_0}[\delta'], \quad (\sigma')^{a_1\dots a_{k'}} \overset{\tilde{\iota}}{\mapsto} \mathbb{Y}^{\ a_1}_{\mathbf{A_1}} \cdots \mathbb{Y}^{\ a_{k'}}_{\mathbf{A_{k'}}} (\sigma')_{a_1\dots a_{k'}}$$

for a chosen metric in the conformal class. Then we apply a suitable polynomial operator in the curved Casimir to obtain a conformally invariant extension $\tilde{F}_{\mathbf{A_1}\dots\mathbf{A_{k'}}} := \tilde{P}(\mathcal{C})(\tilde{\iota}(\sigma')_{A_1\dots A_k}) \in \tilde{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_0}[\delta']$. A similar reasoning as above shows that $\tilde{P}$ has order k' and

$$\tilde{F}_{\mathbf{A_1}\dots\mathbf{A_{k'}}} = \mathbb{Y}_{\mathbf{A_1}a_1} \cdots \mathbb{Y}_{\mathbf{A_{k'}}a_{k'}} \tilde{p}(\delta')(\sigma')^{a_1\dots a_{k'}} + \ldots + \mathbb{W}_{(\mathbf{A_1}} \ldots \mathbb{W}_{\mathbf{A_{k'}})} \nabla_{(a_1} \cdots \nabla_{a_{k'})_0} (\sigma')^{a_1\dots a_{k'}} \tag{17}$$

on $\mathbb{R}^n$ for a polynomial $\tilde{p}$ of the order k'. In this case we need to know $\tilde{p}(\delta')$ explicitly; following [10] we computes

$$\tilde{p}(\delta') = \prod_{i=1}^{k'} (\delta' + n + k' + i - 2).$$

In fact, analogues of this splitting are well–known, see e.g. [23, 6.2.3] or [31, 2.1.4].

In the last step we use the duality between $\bar{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_0}$ and $\tilde{\mathcal{E}}_{(\mathbf{A_1}\dots\mathbf{A_k})_0}$. From this it follows that that $Q^{\sigma'}_{k',0}(f) := \tilde{F}^{\mathbf{A_1}\dots\mathbf{A_{k'}}} \bar{F}_{\mathbf{A_1}\dots\mathbf{A_{k'}}}$ is a conformally invariant bilinear operator. Considering $Q^{\sigma'}_{k',0}$ as a linear operator $\mathcal{E}[w] \to \mathcal{E}[w + \delta']$, it follows from (16) and (17) that

$$\tilde{F}^{\mathbf{A_1}\dots\mathbf{A_{k'}}} \bar{F}_{\mathbf{A_1}\dots\mathbf{A_{k'}}} = \tilde{p}(\delta')(\sigma')^{a_1\dots a_{k'}} \nabla_{(a_1} \cdots \nabla_{a_{k'})_0} f + lot$$

where "*lot*" denotes the lower order terms.

It remains to verify the strong invariance of $Q^{\sigma'}_{k',0}(f)$. But this follows from the fact that the curved Casimir is a strongly invariant linear differential operator. □

Remark. The formula for the curved Casimir operator can be easily given explicitly via tractors. First we define put $\mathbb{H}_{\mathbf{AB}} := h_{A^1B^1}h_{A^2B^2}$ where we skew over $[A^1A^2] = \mathbf{A}$ (hence also over $[B^1B^2] = \mathbf{B}$). Since $\mathcal{E}_{\mathbf{B}}$ is the adjoint tractor bundle, there is an inclusion $\mathcal{E}_{\mathbf{B}} \hookrightarrow \mathrm{End}(\mathcal{T})$ for any tractor bundle $\mathcal{T}$. This yields also $\mathcal{E}_{\mathbf{AB}} \hookrightarrow \mathcal{E}_{\mathbf{A}} \otimes \mathrm{End}(\mathcal{T})$, the image of $\mathbb{H}_{\mathbf{AB}}$ under this inclusion will be denoted by $\mathbb{H}_{\mathbf{A}} \in \mathcal{E}_{\mathbf{A}} \otimes \mathrm{End}(\mathcal{T})$. If $F \in \mathcal{T}$, the application of this endomorphism will be denoted by $\mathbb{H}_{\mathbf{A}}\sharp F \in \mathcal{E}_{\mathbf{A}} \otimes \mathcal{T}$. Explicitly, $\mathbb{H}_{\mathbf{A}}\sharp F_C = \mathbb{H}_{\mathbf{A}C}{}^P F_P$ for $\mathcal{T} = \mathcal{E}_C$ and the general case $\mathcal{T} \subseteq (\bigotimes \mathcal{E}_C) \otimes \mathcal{E}[w]$ is given by the Leibnitz rule. (We put $\mathbb{H}_{\mathbf{A}}\sharp$ to be trivial on $\mathcal{E}[w]$.)

If $\mathcal{T}$ is a tractor bundle then the differential operator

$$\mathcal{D}_{\mathbf{A}} : \mathcal{T} \otimes \mathcal{E}[w] \to \mathcal{T} \otimes \mathcal{E}_{\mathbf{A}}[w], \quad \mathcal{D}_{\mathbf{A}} := w\mathbb{W}_{\mathbf{A}} + \mathbb{X}^a_{\mathbf{A}}\nabla_a + \mathbb{H}_{\mathbf{A}}\sharp$$

is the (conformally invariant) *fundamental derivative* [7] up to a nonzero scalar multiple. The curved Casimir $\mathcal{C}$ is defined as $\mathcal{C} := \mathcal{D}^{\mathbf{A}}\mathcal{D}_{\mathbf{A}} : \mathcal{T} \otimes \mathcal{E}[w] \to \mathcal{T} \otimes \mathcal{E}[w]$. The explicit formula for $\mathcal{C}$ in terms of a chosen Levi–Civita connection from the conformal class can be easily obtained from the previous display.

3.2. **The general case $Q_{k',\ell}$.** Recall $k', \ell \geq 0$, $(\sigma')^{a_1 \dots a_{k'}} \in \mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta']$ and $f \in \mathcal{E}[w]$, $\delta', w \in \mathbb{R}$. We shall construct $Q_{k',\ell}$ by an inductive procedure. The main step is the construction of $Q^{\sigma'}_{k',\ell+1}$ from $Q^{\sigma'}_{k',\ell}$.

Proposition 3.2. *Fix $\delta' \in \mathbb{R}$ and assume there is an explicit construction of the quantization $Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2\ell]$, $k', \ell \geq 0$ with the leading term $\sigma^{a_1 \dots a_{k'}}\nabla_{a_1} \dots \nabla_{a_{k'}}\Delta^\ell$ for every $w \in \mathbb{R}$. Also assume $Q_{k',\ell}$ is strongly invariant in the sense of Theorem 3.1. Then*

$$\widetilde{Q}^{\sigma'}_{k',l} := D^B Q^{\sigma'}_{k',\ell} D_B : \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2(\ell + 1)],$$
$$\widetilde{Q}^{\sigma'}_{k',\ell}(f) = -(\delta' - \ell)(n + 2\delta' + 2(k' - \ell) - 2)\sigma^{a_1 \dots a_{k'}}\nabla_{a_1} \dots \nabla_{a_{k'}}\Delta^{\ell+1} + lot$$

for every $w \in \mathbb{R}$. Here "lot" denotes lower order terms.

The operator $\widetilde{Q}_{k',l} : \mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'] \times \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2(\ell + 1)]$ is a conformally invariant bilinear operator. Moreover, it is strongly invariant in the sense of Theorem 3.1. We put $Q^{\sigma'}_{k',l+1} := \widetilde{Q}^{\sigma'}_{k',l}$.

Proof. We shall start with the discussion on the invariance. Since $Q_{k',\ell} : \mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'] \times \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2\ell]$ is assumed to be strongly invariant (in the sense of Theorem 3.1), it is also invariant as $Q_{k',l} : \mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'] \times \mathcal{E}_B[w] \to \mathcal{E}[w + \delta' - 2\ell]$. Therefore the composition

$$\mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'] \times \mathcal{E}[w] \xrightarrow{\mathrm{id} \times D_B} \mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'] \times \mathcal{E}_B[w-1] \xrightarrow{Q_{k',\ell}} \mathcal{E}[(w-1) + \delta' - 2\ell] \xrightarrow{D^B} \mathcal{E}[w + \delta' - 2\ell - 2]$$

is a conformally invariant bilinear operator. The strong invariance of $\widetilde{Q}_{k',l}$ follows from the strong invariance of D^B.

It remains to compute the leading symbol of $\widetilde{Q}^{\sigma'}_{k',\ell}$, we shall do it by a direct computation. The operator D^B is explicitly given by the sum of three terms on the right hand side of (10). Decomposing both application of tractor D in the formula for $\widetilde{Q}^{\sigma'}_{k',l}$ accordingly, we obtain overall 9 leading terms. Note $Q^{\sigma'}_{k',\ell} D_B f =$

$\big[(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}+lot\big]\Delta^\ell D_B f\in\mathcal{E}_B[w']$ where $f\in\mathcal{E}[w]$ and $w'=w+\delta'-2\ell-1$.

Although the tractor D is of the second order and $Q^{\sigma'}_{k',\ell}$ is of the order $k'+2\ell$, the leading term of $\widetilde{Q}^{\sigma'}_{k',\ell}$ turns out to have order $k'+2\ell+2$ in the generic case. (We use the tractor D twice so one might expect the order $k'+2\ell+4$.) To show this we will collect all terms of the order at least $k'+2\ell+2$. In fact, we shall do this in details only for the leading term $(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell$ of $Q^{\sigma'}_{k',\ell}$. But it will be obvious from the form of all 9 summands this is sufficient. Below we shall use $lot_{\leq o}$ to denote terms of the order at most o, $lot_{<o}$ will denotes cases of order smaller than o. To simplify the notation we will henceforth work with the Euclidean metric; then all terms on the right hand side of (12) and (13) denoted by "ct" vanish.

We shall start with $w'(n+2w'-2)Y^B Q^{\sigma'}_{k',\ell}D_B f$; decomposing D_B here according to (10) yields first three summands. The first one is

$$\begin{aligned}&w'(n+2w'-2)Y^B Q^{\sigma'}_{k',\ell}\big[w(n+2w-2)Y_B f\big]=\\&=w'(n+2w'-2)w(n+2w-2)Y^B(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell Y_B f=0.\end{aligned}\tag{18}$$

The reason is that the tractor Y^B contracts nontrivially only with X_B according to (8) and X_B appear on the right hand side of $\nabla_{(a_1}\dots\nabla_{a_{k'})_0}\Delta^\ell Y_B f$ according to (12) and (13) involves curvature. Analogously we obtain

$$w'(n+2w'-2)Y^B Q^{\sigma'}_{k',\ell}[(n+2w-2)Z^b_B\nabla_b f]=0.\tag{19}$$

Looking at the X_B-terms of $Q^{\sigma'}_{k',\ell}(-X_B\Delta f)$, we see from (12) and (13) that

$$\begin{aligned}&w'(n+2w'-2)Y^B Q^{\sigma'}_{k',\ell}[-X_B\Delta f]=\\&=-w'(n+2w'-2)(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}f+lot_{\leq k'+2\ell+1}.\end{aligned}\tag{20}$$

Next we shall compute $(n+2w'-2)Z^{Bb}\nabla_b Q^{\sigma'}_{k',\ell}D_B f$, we obtain again three summands. This is contraction of $(n+2w'-2)Z^b_B$ with

$$\begin{aligned}\nabla_b Q^{\sigma'}_{k',\ell}D_B f=&\big[(\nabla_b(\sigma')^{a_1\dots a_{k'}})\nabla_{a_1}\dots\nabla_{a_{k'}}+(\sigma')^{a_1\dots a_{k'}}\nabla_b\nabla_{a_1}\dots\nabla_{a_{k'}}\\&+lot_{<k'}\big]\Delta^\ell D_B f.\end{aligned}$$

We need to discuss here only the first two terms in the square bracket here and only $Z^{\bar b}_B$-terms according to (8). First, it is easy to see that

$$(n+2w'-2)Z^{Bb}(\nabla_b(\sigma')^{a_1\dots a_{k'}})\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell D_B f=lot_{\leq k'+2\ell+1}.$$

(The component $w(n+2w-2)Y_B$ of D_B does not contribute to the right hand side of the previous display at all and the remaining components $(n+2w-2)Z^{\bar b}_B\nabla_{\bar b}$ and $-X_B\Delta$ contribute by terms of the equal $\leq k'+2\ell+1$.) Hence it remains to collect $Z^{\bar b}_B$-terms of $(\sigma')^{a_1\dots a_{k'}}\nabla_b\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell D_B f$. Applying (10) to D_B, we

obtain three more summands. A short computation reveals that

$$(n+2w'-2)Z^{Bb}(\sigma')^{a_1\dots a_{k'}}\nabla_b\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell\big[w(n+2w-2)Y_Bf\big]=0, \tag{21}$$

$$\begin{aligned}(n+2w'-2)&Z^{Bb}(\sigma')^{a_1\dots a_{k'}}\nabla_b\nabla_{a_{1'}}\dots\nabla_{a_{k'}}\Delta^\ell\big[(n+2w-2)Z_B^{\bar b}\nabla_{\bar b}f\big]=\\ &=(n+2w'-2)(n+2w-2)Z^{Bb}(\sigma')^{a_1\dots a_{k'}}\nabla_bZ_B^{\bar b}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell\nabla_{\bar b}f=\\ &=(n+2w'-2)(n+2w-2)\sigma^{a_1\dots a_{k'}}\Delta^{\ell+1}f\end{aligned} \tag{22}$$

$$\begin{aligned}(n+2w'-2)&Z^{Bb}\sigma^{a_1\dots a_{k'}}\nabla_b\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell\big[-X_B\Delta f\big]=\\ &=-(n+2w'-2)Z^{Bb}\sigma^{a_1\dots a_{k'}}\nabla_b\nabla_{a_1}\dots\nabla_{a_{k'}}\big[2\ell Z_B^{\bar b}\nabla_{\bar b}\Delta^\ell+X_B\Delta^{\ell+1}\big]f=\\ &=-(n+2w'-2)Z^{Bb}(\sigma')^{a_1\dots a_{k'}}\nabla_b\big[X_B\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}\\ &\qquad+Z_B^{\bar b}\big(2\ell\nabla_{a_1}\dots\nabla_{a_{k'}}\nabla_{\bar b}\Delta^\ell+k'\boldsymbol{g}_{\bar b a_1}\nabla_{a_2}\dots\nabla_{a_{k'}}\Delta^{\ell+1}\big)\big]f=\\ &=-(n+2w'-2)(2\ell+k'+n)(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}f.\end{aligned} \tag{23}$$

Beside the fact that Z^{Bb} contracts nontrivially only with $Z_B^{\bar b}$, we have used (13) to commute Δ^ℓ with $Z_B^{\bar b}$, (12) to commute $\nabla_{a_1}\dots\nabla_{a_{k'}}$ with $Z_B^{\bar b}$ and (11) to commute ∇_b with $Z_B^{\bar b}$.

It remains to compute $-X^B\Delta Q^{\sigma'}_{k',\ell}D_Bf$. The computation is analogous to previous cases but getting more tedious. First we observe

$$\begin{aligned}-X^B\Delta Q^{\sigma'}_{k',\ell}D_Bf=-X^B\big[&(\Delta(\sigma')^{a_1\dots a_{k'}})\nabla_{a_1}\dots\nabla_{a_{k'}}+2(\nabla^p(\sigma')^{a_1\dots a_{k'}})\nabla_p\nabla_{a_1}\dots\nabla_{a_{k'}}\\ &+(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta+lot_{\le k'-1}\big]\Delta^\ell D_Bf.\end{aligned}$$

We shall discuss only the first three terms in the square bracket here. One can compute that

$$-X^B\big[(\Delta(\sigma')^{a_1\dots a_{k'}})\nabla_{a_1}\dots\nabla_{a_{k'}}+2(\nabla^p(\sigma')^{a_1\dots a_{k'}})\nabla_p\nabla_{a_1}\dots\nabla_{a_{k'}}\big]\Delta^\ell D_Bf=lot_{\le k'+2\ell+1}$$

so it remains to compute only $-X^B(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}D_Bf$. This yields three summands according to (10). After some computation we obtain

$$\begin{aligned}-&X^B(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}\big[w(n+2w-2)Y_Bf\big]=\\ &=-w(n+2w-2)(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}f,\end{aligned} \tag{24}$$

$$\begin{aligned}-&X^B(\sigma')^{a_1\dots a'_k}\nabla_{a_1}\dots\nabla_{a'_k}\Delta^{\ell+1}\big[(n+2w-2)Z_B^{\bar b}\nabla_{\bar b}f\big]=\\ &=-(n+2w-2)X^B(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\\ &\qquad\big[-2(\ell+1)Y^B\nabla^{\bar b}\Delta^\ell\nabla_{\bar b}+Z_B^{\bar b}\nabla_{\bar b}\Delta^{\ell+1}\big]f=\\ &=-(n+2w-2)\big[-2(\ell+1)-k'\big](\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}f,\end{aligned} \tag{25}$$

$$\begin{aligned}-&X^B(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}\big[-X_B\Delta f\big]=X^B\sigma^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\\ &\qquad\big[-(\ell+1)(n+2\ell)Y_B\Delta^{\ell+1}+2(\ell+1)Z_B^{\bar b}\nabla_{\bar b}\Delta^{\ell+1}+X_B\Delta^{\ell+2}\big]f=\\ &=\big[-(\ell+1)(n+2\ell)-2k'(\ell+1)\big](\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}f.\end{aligned} \tag{26}$$

The last step of the proof is to sum up the right hand sides of 9 relations (18), (19), (20), (21), (22), (23) and (24), (25), (26) above. That is, we need to compute

the scalar

$$-w'(n+2w'-2)+(n+2w'-2)(n+2w-2)-(n+2w'-2)(2\ell+k'+1)$$
$$-w(n+2w-2)+(n+2w-2)(2\ell+k'+2)-(\ell+1)(n+2\ell+2k')$$

where $w'=w+\delta'-2\ell-1$. This requires some work, the result is $-(\delta'-\ell)(n+2\delta'+2k'-2\ell-2)$ and the proposition follows. Note the resulting scalar does not depend on w; this is a good verification that the computations throughout the proof are correct. $\square$

Theorem 3.3. *Let $k',\ell\geq 0$, $(\sigma')^{a_1\ldots a_{k'}}\in\mathcal{E}^{(a_1\ldots a_{k'})_0}[\delta']$ and $f\in\mathcal{E}[w]$, $\delta',w\in\mathbb{R}$. Then*

$$Q^{\sigma'}_{k',\ell}:=D^{B_1}\cdots D^{B_{k'}}Q^{\sigma'}_{k',0}D_{B_{k'}}\cdots D_{B_1}:\mathcal{E}[w]\to\mathcal{E}[w+\delta'-2\ell]$$

defines the conformally invariant quantization with the leading term $(\sigma')^{a_1\ldots a_{k'}}\nabla_{a_1}\ldots\nabla_{a_{k'}}\Delta^\ell$ (up to a sign) for every weight δ' satisfying

$$\delta'\notin\Sigma_{k',\ell}:=\Sigma_{k',0}\cup\Sigma'_{k',\ell}\cup\Sigma''_{k',\ell} \tag{27}$$

where $\Sigma_{k',0}$ is given by (15),

(28)

$$\Sigma'_{k',\ell}=\{(j-1)\mid j=1,\ldots,\ell\},\quad \Sigma''_{k',\ell}=\{-\frac{1}{2}(n+2k'-2j)\mid j=1,\ldots,\ell\}\quad \textit{for } \ell\geq 1.$$

We put $\Sigma'_{k',0}=\Sigma''_{k',0}:=\emptyset$. Moreover, $Q^{\sigma'}_{k',\ell}$ is strongly invariant in the sense of Theorem 3.1.

Proof. The set of critical weights $\Sigma_{k',\ell}$ easily follows (by induction with respect to ℓ) from Proposition 3.2. Since the tractor D and $Q^{\sigma'}_{k',0}$ are strongly invariant, the last claim is obvious. $\square$

Remark. Let us note the previous theorem yields an inductive formula for the conformal quantization as $Q^{\sigma'}_{k',\ell+1}=D^BQ^{\sigma'}_{k',\ell}D_B$. Similarly, we can describe the set of critical weights inductively as $\Sigma_{k',\ell+1}=\Sigma_{k',\ell}\cup\{\ell,-\frac{1}{2}(n+2k'-2\ell-2)\}$ where $\Sigma_{k',0}$ is given by (15).

4. Critical weights

We shall discuss the cases $\delta'\in\Sigma_{k',\ell}$ from (27) in detail. First, a simple calculation shows

Lemma 4.1. *(i) $2\ell\notin\Sigma_{k',\ell}$ for all $k',\ell\geq 0$.*
(ii) The sets $\Sigma_{k',0}$ and $\Sigma'_{k',\ell}\cup\Sigma''_{k',\ell}$ are disjoint. $\square$

The symbols of the quantization $\mathcal{E}[w]\to\mathcal{E}[w]$ (i.e. with zero shift) are of a special interest [12]. The flat quantization developed there is never critical for such symbols [12, 3.1]. The previous lemma (i) recovers this observation for the curved quantization $Q^{\sigma'}_{k',\ell}$.

The critical weights are closely related to existence to natural linear conformal operators. They are completely classified in the locally flat case [2, (3.1)] (or see the summary in [17, Section 3]). Using this we obtain

Proposition 4.2. *Assume the manifold M is conformally flat. If $\delta' \in \Sigma_{k',\ell}$ then there exists a nontrivial natural linear conformal operator on $\mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta']$ as follows*

$$\begin{aligned}
&\mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'] \longrightarrow \mathcal{E}^{(a_1 \dots a_{i-1})_0}[\delta'], && \delta = -(n+k'+i-2) \in \Sigma_{k',0},\\
&\mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'] \longrightarrow \mathcal{E}^{(a_1 \dots a_{k'+j})_0}[\delta'-2j], && \delta' = j-1 \in \Sigma'_{k',\ell},\\
&\mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'] \longrightarrow \mathcal{E}^{(a_1 \dots a_{k'})_0}[\delta'-2j], && \delta' = -\frac{1}{2}(n+2k'-2j) \in \Sigma''_{k',\ell}.
\end{aligned}$$

The case $\delta' \in \Sigma_{k',0}$ is a divergence type operator of the order $k'-i+1$, $\delta' \in \Sigma'_{k',\ell}$ is the conformal Killing operator of the order j and $\delta' \in \Sigma''_{k',\ell}$ yields a Laplacian type operator of the order $2j$. Note this operator is not unique as generally $\Sigma'_{k',\ell} \cap \Sigma''_{k',\ell} \neq \emptyset$.

The operator $Q^{\sigma'}_{k',\ell}$ does not provide a conformally invariant quantization for $\delta' \in \Sigma_{k',\ell}$. Such a quantization can exists, though, for certain w, as observed in lower order cases [13, 11]. (Note it is not unique even in the flat case then.) Assuming $\delta' \in \Sigma_{k',\ell}$, we shall find such w for all k', ℓ in the flat setting; the curved case is more involved. In particular, it is closely related to existence of natural linear conformal operators

$$\begin{aligned}
&S_p : \mathcal{E}[p-1] \longrightarrow \mathcal{E}_{(a_1 \dots a_p)_0}[p-1], && L_p : \mathcal{E}[-n/2+p] \longrightarrow \mathcal{E}[-n/2-p],\\
&S_p(f) = \nabla_{(a_1} \dots \nabla_{a_p)_0} f + lot, && L_p(f) = \Delta^p f + lot,
\end{aligned}$$

for $p \geq 1$ (so p is *not* an abstract index here). If n is odd or M is conformally flat, these operators exist for all $p \geq 1$. In the curved case for n even, S_p exists for all $p \geq 1$ and L_p exists for $1 \leq p \leq n$, see [9, 22, 20]. They are strongly invariant (can be given by a strongly invariant formula) in the flat case; in the curved case, S_p is strongly invariant always and L_p only for $p < n$.

Theorem 4.3. *Assume $\delta' \in \Sigma_{k',\ell}$ and $f \in \mathcal{E}[w]$. Then there is always a choice of $w \in \mathbb{R}$ for which there is a quantization $Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w+\delta]$ with the leading term $(\sigma')^{a_1 \dots a_{k'}} \nabla_{a_1} \dots \nabla_{a_{k'}} \Delta^\ell f$ in the flat case. This is true also on curved manifols under an additional assumption $\ell \leq n$.*

Explicitly, the quantization is given by formulae

$$\begin{aligned}
&Q^{\sigma'}_{k',0} L_\ell : \mathcal{E}[-n/2+\ell] \to \mathcal{E}[\delta'-n/2-\ell], && \delta' \in \Sigma'_{k',\ell} \cup \Sigma''_{k',\ell}\\
&D^{B_1} \cdots D^{B_\ell} \iota(\sigma') S_{k'} D_{B_1} \cdots D_{B_\ell} : \mathcal{E}[k'+\ell-1] \to \mathcal{E}[\delta'+k'-\ell-1], && \delta' \in \Sigma_{k',0}
\end{aligned}$$

where $\iota(\sigma')$ is the complete contraction of the image of S_p with σ'.

Proof. The conformal invariance is obvious (recall $S_{k'}$ has the source space $\mathcal{E}[k'-1]$ and is strongly invariant). It remains to verify the displayed operators have the required leading term (up to a nonzero multiple). In the case $\delta' \in \Sigma'_{k',\ell} \cup \Sigma_{k',\ell}$, this follows from the leading term of L_ℓ, properties of $Q^{\sigma'}_{k',0}$ in Theorem 3.3 and Lemma 4.1 (ii).

Assume $\delta' \in \Sigma_{k',0}$ and denote by $\overline{Q}^{\sigma'}_{k',\ell}$ the displayed operator for such δ'. We need to compute the leading term of $\overline{Q}^{\sigma'}_{k',\ell}$. Observe the generic quantization $Q^{\sigma'}_{k',\ell}$

is constructed in a similar way as $\overline{Q}^{\sigma'}_{k',\ell}$ – only the subfactor $Q^{\sigma'}_{k',0}$ of $Q^{\sigma'}_{k',\ell}$ (see the display in Theorem 3.3) is replaced by $\iota(\sigma')S_{k'}$ in $\overline{Q}^{\sigma'}_{k',\ell}$. It is mentioned in the proof of Proposition 3.2 that only the term $(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell$ of $Q^{\sigma'}_{k',\ell}$ contributes to the generic leading term $(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^{\ell+1}$ of $\widetilde{Q}^{\sigma'}_{k',\ell}$, see Proposition 3.2 for the notation. However $\iota(\sigma')S_{k'}$ has the leading term $(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}$ for $\delta'\in\Sigma_{k',0}$ as well as $Q^{\sigma'}_{k',0}$ for $\delta'\notin\Sigma_{k',0}$. It follows that $\overline{Q}^{\sigma'}_{k',\ell}$ has $(\sigma')^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell$ as the leading term for all $\delta'\in\Sigma'_{k',\ell}\cup\Sigma''_{k',\ell}$. Using Lemma 4.1(ii) the theorem follows. □

Remark. As expected, the quantization used in the previous Proposition is not unique. If, for example, $\delta'\in\Sigma'_{k',\ell}\cup\Sigma''_{k',\ell}$ but $\delta'\notin\Sigma'_{k',\ell_0}\cup\Sigma''_{k',\ell_0}$ for some $\ell_0<\ell$, one can use also the operator $Q^{\sigma'}_{k',\ell_0}L_{\ell-\ell_0}$ which is invariant on $\mathcal{E}[-n/2+\ell-\ell_0]$. (A similar idea can be used for $\delta'\in\Sigma_{k',0}$.)

5. Comparision with related results

There are several related results concerning conformal quantization either in the flat case [12] or in lower order curved cases [13, 11, 26]. On the other hand, conformal quantization is a special case of bilinear operators constructed in [23]. We discuss [23] and [12] in more details here.

In the groundbreaking Kroeske's thesis [23], a general construction of invariant bilinear operators (or "invariant pairing") for (curved) parabolic geometries is developed. However, we require that $Q^{\sigma'}_{k',\ell}:\mathcal{E}[w]\to\mathcal{E}[w+\delta']$ is defined for every $w\in\mathbb{R}$ whereas the construction in [23] generally yields *couples* of critical weights (δ',w). Considering the conformal case, it is probably possible to obtain the quantization on densities from the detailed exposition in [23, Section 5] with some set of critical weights. Our construction of the bilinear operator Q is much simpler as it is designed to the special case needed here.

In [12], the study of conformal quantization was initiated. The set of critical weights (as a subset of "resonant" weights) agrees with our result up to the order 2. In the order 3, also the critical case $\mathcal{E}^{(a_1a_2a_3)}[\delta]$ where $\delta=-\frac{2}{3}(n+2)$. (Note we have used $\mathcal{E}[-nw]=\mathcal{F}_w$ (cf. the introduction) to pass to our notation. In fact, the value $\frac{2(n+2)}{3n}$ is obtained by the choice $(k,l,s,t)=(3,0,1,0)$ in [12, (3.7)], see also [12, Theorem 3.5,3.6].) $\mathcal{E}^{(a_1a_2a_3)}[\delta]$ has two irreducible components, in particular $\mathcal{E}^{(a_1a_2a_3)_0}[\delta]$ and $\mathcal{E}^a[\delta+2]$. However $\delta\notin\Sigma_{3,0}$ and $\delta+2\notin\Sigma_{1,1}$ for $\delta=-\frac{2}{3}(n+2)$ and generic n. Note there is no nontrivial natural linear flat conformal operator on $\mathcal{E}^a[\delta+2]$ or $\mathcal{E}^{(a_1a_2a_3)_0}[\delta]$ in generic dimensions.

It seems plausible the critical set $\Sigma_{k',\ell}$ is minimal for conformal quantization with the corresponding leading term. Although no non-existence results for higher orders are known (up to our knowledge), we conjecture that if $(\sigma')^{a_1\dots a_{k'}}\in\mathcal{E}^{(a_1\dots a_{k'})_0}[\delta']$, $\delta'\in\Sigma_{k',\ell}$ then there is no conformal quantization $\mathcal{E}[w]\to\mathcal{E}[w+\delta'-2\ell]$ with the leading term $\sigma^{a_1\dots a_{k'}}\nabla_{a_1}\dots\nabla_{a_{k'}}\Delta^\ell$ for generic $w\in\mathbb{R}$. The minimality is closely related to Proposition 4.2 and Theorem 4.3. In particular, we expect a version of Proposition 4.2 to be a necessary condition for nonexistence.

6. Examples

The tractor formulae are easily rewritten to the usual formulae in the Levi–Civita covariant derivative and its curvature. We shall demonstrate this on the quantization of the order three (which in fact known [11]). There are two irreducible leading terms:

Example 6.1. We shall start with the case $(\sigma')^{abc}\nabla_a\nabla_b\nabla_c$ where $(\sigma')^{abc} \in \mathcal{E}^{(abc)_0}[\delta']$. We shall avoid the general result from [23] as one can directly verify the differential operators

$$\begin{aligned}(\sigma')^{abc} \mapsto M^{ABC}_{\,a\,b\,c}(\sigma')^{abc} := &(n+\delta'+2)(n+\delta'+3)(n+\delta'+4)Z^A_a Z^B_b Z^C_c(\sigma')^{abc}\\ &- 3(n+\delta'+2)(n+\delta'+3)X^{(A}Z^B_b Z^{C)}_c\nabla_p(\sigma')^{pbc}\\ &+ 3(n+\delta'+3)X^{(A}X^B Z^{C)}_c\big(\nabla_p\nabla_q + (n+\delta'+4)P_{pq}\big)(\sigma')^{pqc}\\ &- X^A X^B X^C\big[\nabla_p\big(\nabla_q\nabla_r + (n+\delta'+4)P_{qr}\big) + 2(n+\delta'+3)P_{pq}\nabla_r\big](\sigma')^{pqr}\end{aligned}$$

and

$$\begin{aligned}f \mapsto \widetilde{D}_{ABC}f := &w(w-1)(w-2)Y_AY_BY_Cf\\ &+ 3(w-1)(w-2)Y_{(A}Y_BZ^c_{C}\nabla_cf\\ &+ 3(w-2)Y_{(A}Z^{(b}_BZ^{c)_0}_{C)}\big(\nabla_b\nabla_c + wP_{bc}\big)\\ &+ Z^{(a}_{(A}Z^b_BZ^{c)_0}_{C)}\big[\nabla_a(\nabla_b\nabla_c + wP_{bc}) + 2(w-1)P_{bc}\nabla_a\big]f\end{aligned}$$

where $f \in \mathcal{E}[w]$. One easily verifies directly they are conformally invariant. (Note the M is a special case of the "middle operator" from [31, 2.1.4].).

The target space of the operator $M^{ABC}_{\,a\,b\,c}$ is a subbundle of $\mathcal{E}^{(ABC)}[\delta'+3]$ and target space of $\widetilde{D}_{ABC}$ is a quotient of $\mathcal{E}_{(ABC)}[w-3]$. These two target spaces are dual to each other via the tractor metric h. Thus the contraction

$$\begin{aligned}f \mapsto \big(M^{ABC}_{\,a\,b\,c}(\sigma')^{abc}\big)\widetilde{D}_{ABC}f = &(n+\delta'+2)(n+\delta'+3)(n+\delta'+4)\\ &(\sigma')^{abc}[\nabla_a(\nabla_b\nabla_c + wP_{bc}) + 2(w-1)P_{bc}\nabla_a]f + lot\end{aligned}$$

where *lot* means "lower order terms", is conformally invariant. Note the we see directly from (8) which terms of $M^{ABC}_{\,a\,b\,c}(\sigma')^{abc}$ and $\widetilde{D}_{ABC}f$ contract nontrivially with each other.

Example 6.2. Another possible leading term in the third order is $(\sigma')^b\nabla_b\Delta$, $(\sigma')^b \in \mathcal{E}^b[\delta']$. Also this case can be solved without the general formula in Theorem 3.3. Similarly as in the previous example, we shall start with the conformally invariant operator $(\sigma')^b \mapsto M^B_b(\sigma')^b := (n+\delta')Z^B_b(\sigma')^b - X^A\nabla_p(\sigma')^p$. Next we apply

the operator D_A and symmetrize over the tractor indices. Overall we obtain

$$
\begin{aligned}
(\sigma')^b \mapsto & M^B_b(\sigma')^b \mapsto D^{(A} M^{B)}_b(\sigma')^b = \delta'(n+\delta')(n+2\delta')Y^{(A}Z^{B)}_b(\sigma')^b \\
& + (n+2\delta')Z^{(A}_a Z^{B)}_b\big[(n+\delta')\nabla^{(a}(\sigma')^{b)_0} + \frac{1}{n}\delta'\boldsymbol{g}^{ab}\nabla_p(\sigma')^p\big] \\
& - \delta'(n+2\delta')X^{(A}Y^{B)}\nabla_p(\sigma')^p \\
& - X^{(A}Z^{B)}_b\big[(n+2\delta'-2)\big(\nabla^b\nabla_p + (n+\delta')P^b{}_p\big)(\sigma')^p + (n+\delta')\big(\Delta+(\delta'+1)J\big)(\sigma')^b\big] \\
& + X^{(A}X^{B)}\big[\big(\Delta+\delta' J\big)\nabla_p + (n+\delta')\big((\nabla_p J) + 2P_{pq}\nabla^q\big)\big](\sigma')^p
\end{aligned}
$$

after some computation using (10). The target space of $D^{(A}M^{B)}_b$ is a subbundle of $\mathcal{E}^{(AB)}[\delta']$. Hence we need an operator which takes $f \in \mathcal{E}[w]$ in to the dual of this target space. Naively, we can use $D_A D_B f$ but this would kill the leading term $(\sigma')^p\nabla_p\Delta$ for $w = -\frac{n}{2}+2$. In fact, the dual (up to a conformal weight) to $D^{(A}M^{B)}_b(\sigma')^b$ is a quotient of $D_A D_B f$ which we denote by $\widetilde{T}_{AB}f$. After some computation, we obtain that

$$
\begin{aligned}
f \mapsto & \widetilde{T}_{AB} f := w(w-1)(n+2w-2)Y_A Y_B f + \\
& + 2(w-1)(n+2w-2)Y_{(A}Z^b_{B)}\nabla_b f \\
& + Z^{\,a}_{(A}Z^b_{B)}\big[(n+2w-2)\big(\nabla_{(a}\nabla_{b)_0} + wP_{(ab)_0}\big) + \frac{2}{n}(w-1)\boldsymbol{g}_{ab}\big(\Delta + wJ\big)\big]f \\
& - 2(w-1)X_{(A}Y_{B)}\big(\Delta + wJ\big)f \\
& - 2X_{(A}Z^b_{B)}\big[\nabla_b\big(\Delta+wJ\big) + (n+2w-2)P_b{}^p\nabla_p\big]f
\end{aligned}
$$

is conformally invariant. Summarizing, we obtain the conformally invariant quantization on $\mathcal{E}[w]$ by

$$
\begin{aligned}
f \mapsto & \big(D^{(A}M^{B)}_b(\sigma')^b\big)T_{AB}f = \\
& = -\delta'(n+\delta')(n+2\delta')(\sigma')^b\big[\nabla_b\big(\Delta+wJ\big) + (n+2w-2)P_b{}^p\nabla_p\big]f + lot
\end{aligned}
$$

where "*lot*" denotes lower order terms.

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